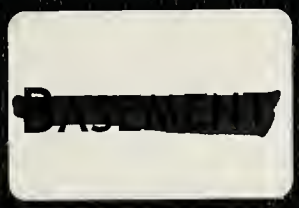

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A Model of Social Insurance with Variable Retirement

P. A. Diamond* and J. A. Mirrlees

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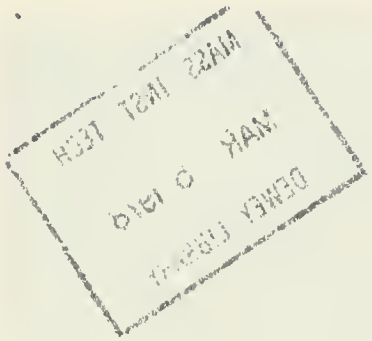
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1. Introduction

No-one knows what work he will be capable of in the future. Uncertainty about earning ability in the last years of life is particularly great. The burden of this risk to the individual is eased both by private insurance and by the tax and social insurance system. Complete relief from risk is not available, because neither private insurance nor public arrangements distinguish fully between low income by choice and low income by necessity. Full insurance might defeat itself through moral hazard. For this reason, the design of optimal social insurance is an interesting and difficult problem. In this paper, we consider a simple form of earning ability risk, and study optimal insurance for a population of identical individuals.[2]

We suppose that everyone lives for the same length of time, and that at any date an individual has either full earning capability or none: there is no partial loss of earning ability. Loss of ability strikes randomly. The government is assumed unable to distinguish those who cannot work from those who merely choose not to. Individuals are assumed to maximize their expected utility, and to be willing, without concern for the truth, to claim inability to work when it suits them.

Three models will be used, with one period, with two periods, and with continuous time. The one-period model allows us to develop simply the basic condition determining whether or not there is a moral hazard problem. For the case where there is a problem, we relate the size of the optimal insurance plan to characteristics of the utility function. With the two-period model, we demonstrate an additional aspect of the moral hazard problem, that taxation of alternative

commodities can increase expected utility. For our model, we find that, under plausible assumptions, an untaxed individual would try to save too much. Thus the optimal social insurance plan needs to be supplemented by an interest income tax. In the continuous-time model, we examine the optimal consumption path while working, and the optimal relationship between pension and date of retirement. Even in the absence of utility-discounting, and of a positive return to saving, we find that it is optimal for consumption to increase with age, and for retirement benefit to increase with retirement age. This leads us to suggest a way of incorporating these conclusions into the benefit structure of the U.S. Social Security system.[3]

2. The One-Period Model

Either the individual is capable of work or he is not. He knows the probability θ that he can work. He is an expected-utility maximizer, his utility being specified by three utility functions:-

$u_1(c)$ = utility of consumption c when working,

$u_2(c)$ = utility when able to work, but not working,

$u_3(c)$ = utility when unable to work.

When working, he produces one unit of output. Assuming nonlinear taxation and one type of individual, the lack of continuous adjustment of labour supply is not critical.

On the assumption that work is unpleasant in the aggregate, we have

$$u_2(c) > u_1(c) \quad \text{for all} \quad c \qquad (1)$$

We also assume that work plus consumption is preferable to no work:

$$u_1(1) > u_2(0) \tag{2}$$

There are no private insurance markets. The government pools risks well enough to be able to plan to distribute consumption, conditional on work, in such a way that the expected value of consumption equals the expected value of output for any individual.

The government being unable to tell why an individual does not work, its policy is entirely described by c_1, consumption for a person who is working, and c_2, consumption for one who is not. This policy provides the individual with expected utility

$$\theta u_1(c_1) + (1 - \theta)u_3(c_2) \quad \text{if he works when he can}$$

$$\theta u_2(c_2) + (1 - \theta)u_3(c_2) \quad \text{if he does not work}$$

Thus he is willing to work if and only if

$$u_1(c_1) \geq u_2(c_2) \tag{3}$$

It is as well to suppose that, even in the case of indifference, the individual works if he is willing and able to do so. We shall comment on this assumption below.

When individuals who can, work, the aggregate resource constraint is

$$\theta c_1 + (1 - \theta)c_2 \leq \theta \tag{4}$$

since the output of a worker is 1. Otherwise, $c_2 = 0$.

The situation can be portrayed in a diagram, with c_1 and c_2 on the axes. This is done in Figures 1 and 2, where the plane

is divided into two regions by the curve $u_1(c_1) = u_2(c_2)$. By assumption (2), the point $c_1 = 1$, $c_2 = 0$ lies below this curve. Everywhere on and below the curve, the individual works if he can; and above it, he does not work. The feasible region is the shaded area plus the origin. Three indifference curves, I_1I_1, I_2I_2, I_3I_3, are drawn, along which expected utility is constant given the private decision on whether to work. It will be noticed that the curve $u_1 = u_2$ lies entirely below the forty-five degree line: this expresses assumption (1).

Evidently there are two cases to consider, which are shown in the two figures. In the first, the indifference curve is tangent to the line with equation $\theta c_1 + (1 - \theta)c_2 = \theta$. Since the slope of the indifference curve is

$$- \frac{\theta}{1 - \theta} \frac{u_1'(c_1)}{u_3'(c_2)} \tag{5}$$

and the slope of the budget line is $-\theta/(1 - \theta)$, the optimum in this case is given by

$$u_1'(c_1^*) = u_3'(c_2^*) \quad \text{and} \quad \theta c_1^* + (1 - \theta)c_2^* = \theta \tag{6}$$

In this case we have the familiar description of a full optimum. The moral hazard constraint is ineffective, as can be seen from the diagram: in the full optimum, people are willing to work if they can.

If at the allocation described by (6)
we have $u_1 < u_2$, the situation is different, and must be as shown in Figure 2 Here the optimum is given by the intersection of the budget constraint and the curve $u_1 = u_2$:

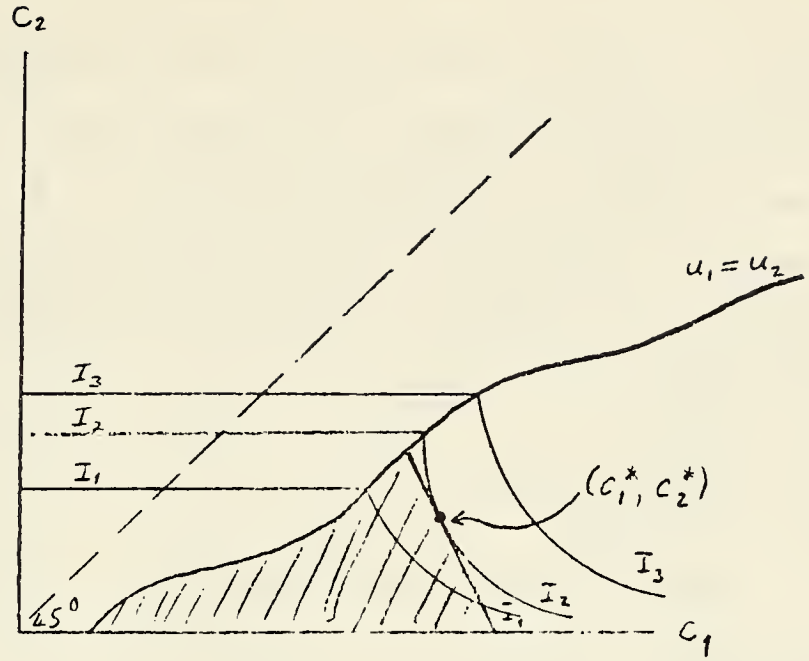

Figure 1

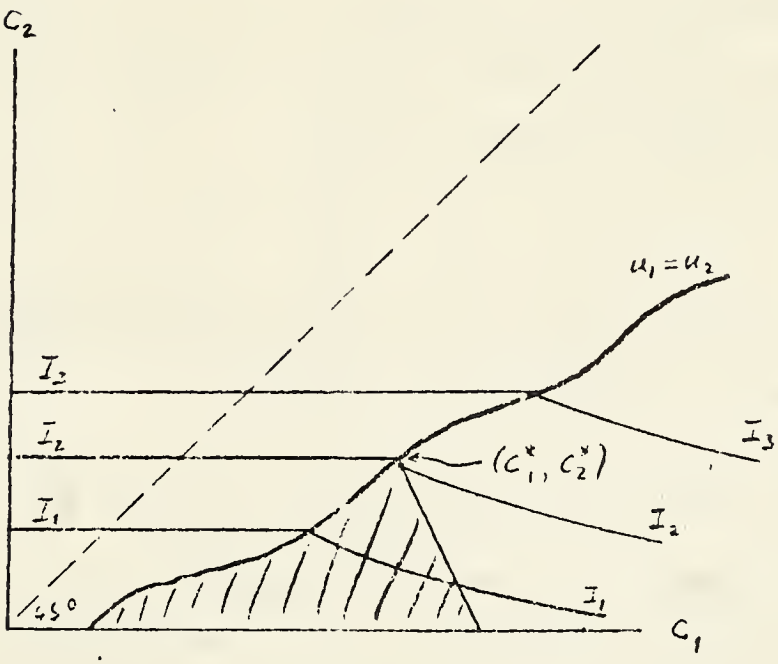

Figure 2

$$u_1(c_1^*) = u_2(c_2^*) \quad \text{and} \quad \theta c_1^* + (1 - \theta)c_2^* = \theta \tag{7}$$

For this case to occur, the indifference curve at this intersection point must slope down less steeply than the budget line. This will be the case if $u_1' \leq u_3'$ at all points of the curve $u_1 = u_2$. If, on the contrary, $u_1' > u_3'$ at all points of the curve, the first case applies. For convenience, we state all this as a formal theorem.[4]

Theorem 1 If for all x and y

$$u_1(x) = u_2(y) \quad \text{implies} \quad u_1'(x) \leq u_3'(y) \tag{8}$$

the optimum is given by (7). If for all x and y

$$u_1(x) = u_2(y) \quad \text{implies} \quad u_1'(x) \geq u_3'(y) \tag{9}$$

the optimum is given by (6).

The conditions in the theorem take on a more interesting shape if we make the further assumption that the loss in the ability to work has an additive effect on utility:

$$u_3(c) = u_2(c) - b, \qquad b \geq 0 \tag{10}$$

Since in this case marginal utility of consumption is the same in the two states, condition (8) can be stated as

$$u_1(x) = u_2(y) \quad \text{implies} \quad u_1'(x) \leq u_2'(y), \quad \text{for all } x \text{ and } y \tag{11}$$

It seems to us that this condition will much more often be relevant than its contrary. For example, it follows when utility is additively separable in consumption and labour, and labour is disliked.

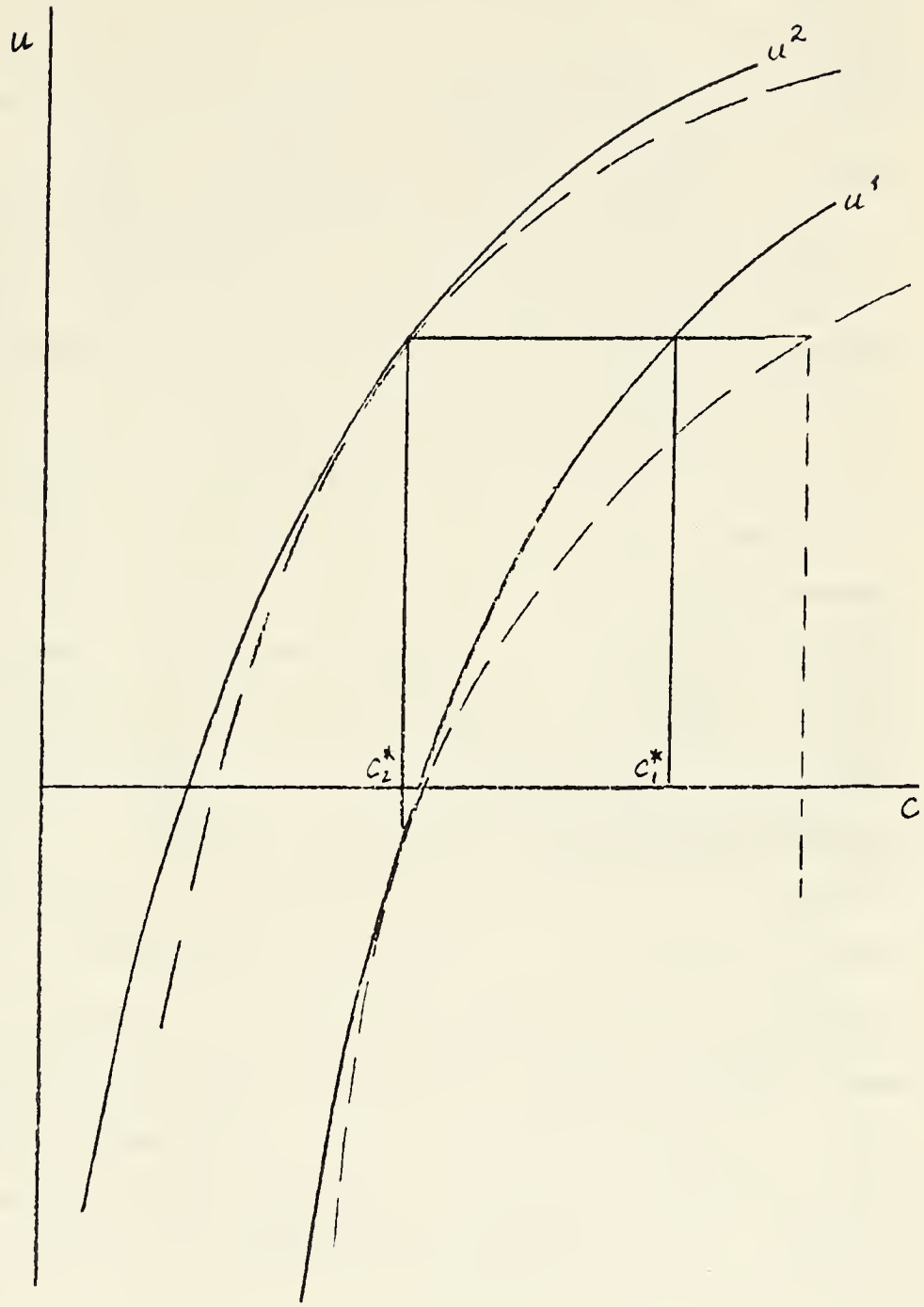

Figure 3

3. Comparative Statics under Moral Hazard

By means of the diagram, we can examine the way in which the optimum changes when probabilities change, when the disutility of labour is changed, and when the utility of consumption is changed.

When θ increases, the budget line becomes steeper, rotating clockwise about the fixed point $(1,0)$. Thus, if moral hazard is present, c_1^* and c_2^* both increase. Similarly an increase in the disutility of work, i.e. $u_2 - u_1$, moves the curve $u_1 = u_2$ downwards, so that c_1^* is increased and c_2^* decreased.

A change in the utility of consumption can also be analysed by considering how it affects the curve $u_1 = u_2$. For example, we can show that an increase in risk-aversion can lead to a reduction in the extent of insurance, which may be contrary to some people's intuition. Suppose utility and marginal utility at c_2^* remain fixed, both with and without work, and that risk-aversion increases. u_1 is thereby decreased for $c > c_2^*$, and in particular for c_1^*. (See Figure 3.) Therefore the curve $u_1 = u_2$ is shifted to the right in the neighbourhood of c_2^*. Consequently the optimal c_1^* is increased and the optimal c_2^* is decreased: the extent of insurance is reduced, as claimed. A different kind of increase in risk-aversion can have the opposite effect.[5] For example, by preserving utility and marginal utility at c_1^* while increasing risk-aversion. By analogy to Figure 3 the curve $u_1 = u_2$ is shifted up.

4. The Two-Period Model

With two periods, we assume that everyone can work in the first

period, and that an individual is able to work in period two with probability θ. These facts are known to the government. We shall want to use this model to discuss the significance of private saving behaviour, but for the present, the government is assumed to offer individuals pairs of first and second period consumptions: (c_o, c_1) if work is done in period two, (c_o, c_2) if not. Utility is now in every case a function of consumption in the two periods. The particular utility function will be denoted by superscript instead of subscript, subscripts being reserved for derivatives.

The budget constraint is taken to be

$$c_o + R[\theta c_1 + (1 - \theta)c_2] = 1 + \theta R \qquad (12)$$

it being assumed that aggregate output when everyone works is unity, and that there is a constant interest factor[6] R. In order to portray the situation on a two-dimensional diagram, we use the budget constraint (12) to eliminate one of the variables when everyone works who is able. It is most convenient to use the variables c_o and

$$z = c_1 - c_2 \qquad (13)$$

where z is the return to working, 1 - z the tax on labour income. Then we have from (12),

$$c_1 = \frac{1 - c_o}{R} + \theta + (1 - \theta)z \qquad (14)$$

$$c_2 = \frac{1 - c_o}{R} + \theta(1 - z) \qquad (15)$$

Since consumption levels have to be nonnegative, it follows that, so long as everyone works when he can, c_o and z are constrained by three linear inequalities, $c_o \geq 0$, $c_1 \geq 0$, and $c_2 \geq 0$.

These define a triangle in (c_o,z)-space.

However, people are willing to work when they can only if

$$u^1(c_o,c_1) \geq u^2(c_o,c_2) \qquad (16)$$

If $z \leq 0$, $c_2 \geq c_1$, so that, maintaining our assumption that work has disutility, $u^2(c_o,c_2) > u^1(c_o,c_1)$. Thus the inequality (16) implies that $z > 0$. Notice also that, since an increase in z increases c_1 and decreases c_2, (c_o,z') satisfies (16) whenever (c_o,z) satisfies (16) and $z' > z$. The region in which people work if they can is the one bounded by ABC in Figure 4. If (c_o,z) is not in this region, either there is no feasible allocation with these values of c_o and z, or no-one works in the second period.

In the latter case, $c_o + Rc_2 = 1$, and expected utility is $\theta u^2(c_o,c_2) + (1 - \theta)u^3(c_o,c_2)$. Let

$$v_m = \text{Max } [\theta u^2 + (1 - \theta)u^3 : c_o + Rc_2 = 1] \qquad (17)$$

be the maximum utility obtainable without work in the second period.

The indifference curves drawn in the diagram are the curves of constant expected utility $\theta u^1 + (1 - \theta)u^3$ on the assumption that work is done in the second period. The case shown is one for which the full optimum - the point F - is not feasible; and the maximum attainable utility is greater than v_m, so that moral hazard is present in the optimum, and work is done in the second period. The same argument as in Section 2 shows that the full optimum is unattainable if

$$u^1(x_o,x_1) = u^2(x_o,x_2) \text{ implies } u^1_1(x_o,x_1) \leq u^3_2(x_o,x_2) \qquad (18)$$

where u_i is the partial derivative with respect to x_i.

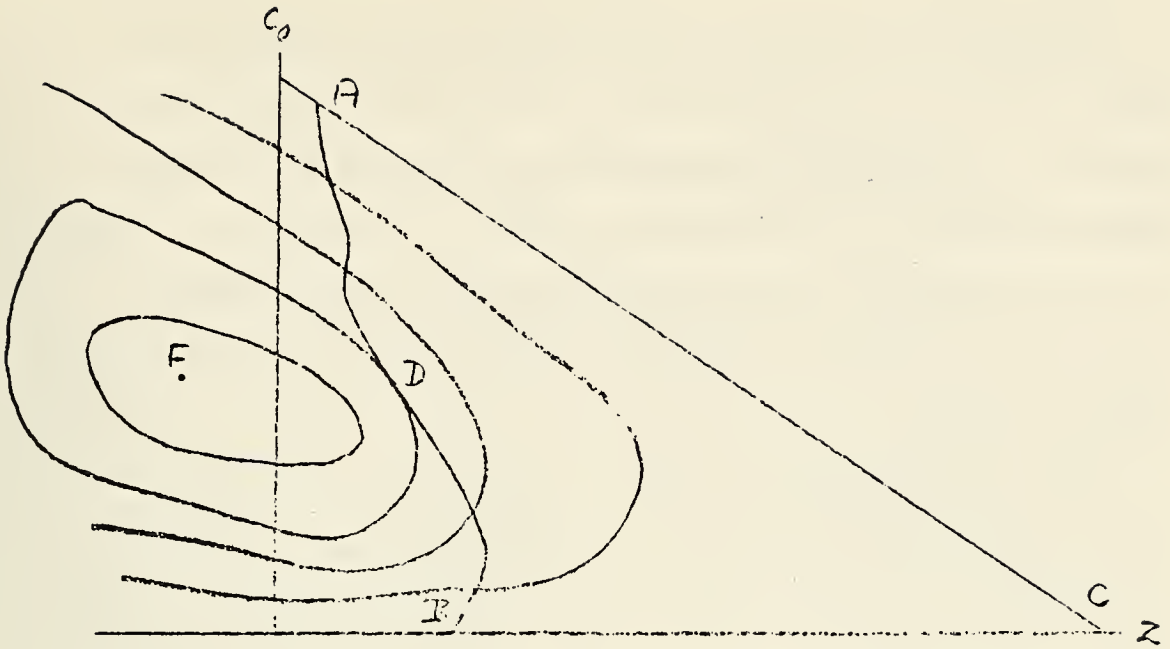

Figure 4

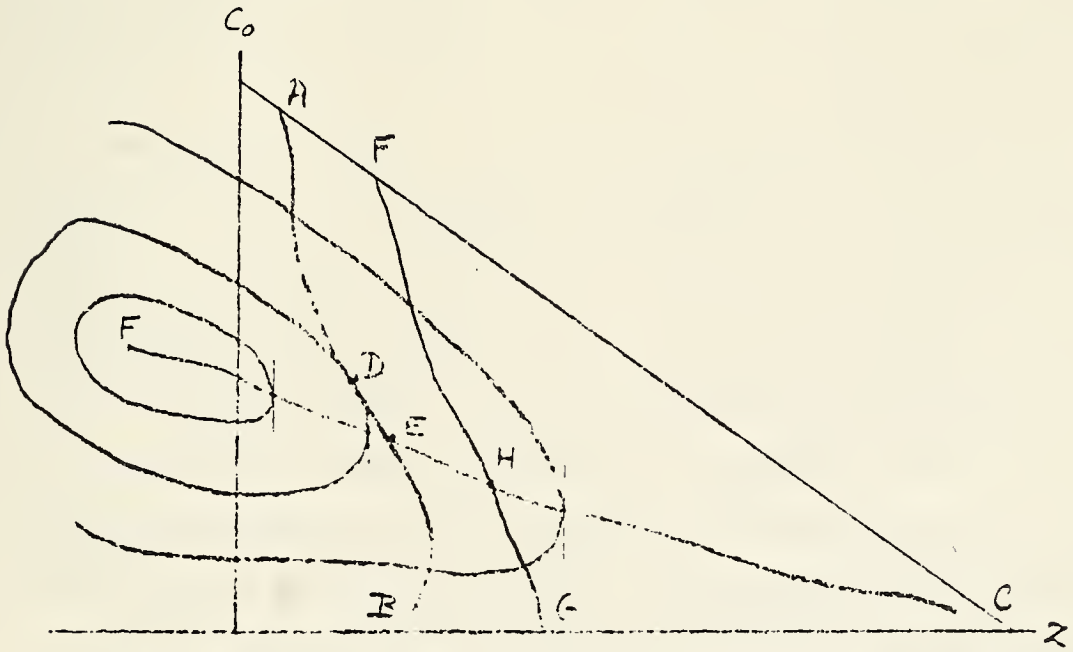

Figure 5

We concentrate on the case where moral hazard is present and there is work in the second period at the optimum. Then at the optimum the indifference curve is tangent to the curve AB. Also a small increase in z would reduce utility. An increase in c_o may either increase or decrease utility, depending on the sign of the slope of the curves at the optimum. We shall discuss conditions for this in a moment.

The interesting point is that a change in c_o that would increase expected utility while z is held constant represents a savings opportunity that the individual would wish to avail himself of if he were allowed to; for, as can be seen from (14) and (15), a change in c_o brings about the changes in c_1 and c_2 that would happen if it were an act of private saving or dissaving. Thus in general, the optimum cannot be obtained without preventing access to the capital market, or, alternatively, imposing an appropriate tax or subsidy on wealth.

It is most likely that the curve AB representing the moral hazard constraint will have a negative slope at the point of tangency. If u^2 tends to minus infinity as c_2 tends to zero, then the curve $u^1 = u^2$ can hit the line $c_2 = 0$ only where c_1 is also zero, i.e. on the vertical axis, with $z = 0$. Since $z > 0$ at the point B, the slope of the curve AB is predominantly negative. We can also find a rather plausible local condition for the optimum to appear as in Figure 4. For this further analysis we assume that u^2 and u^3 differ only by a constant, i.e., have the same derivatives.

Theorem 2 Assume that $u^3 = u^2 - b$,

(i) $u^1(x_o,x_1) = u^2(x_o,x_2)$ implies $u_1^1(x_o,x_1) \leq u_2^2(x_o,x_2)$

(ii) $u^1(x_o,x_1) = u^2(x_o,x_2)$ implies $\left(\frac{\partial x_1}{\partial x_o}\right)_{u^1} \leq \left(\frac{\partial x_2}{\partial x_o}\right)_{u^2}$

Then at the optimum, individuals want to save more.

Proof Our claim is that at the optimum a reduction in c_o would increase expected utility, z being held constant; i.e. that the curve $u^1 = u^2$ has a negative slope at the optimum. Since $u^1 - u^2$ is an increasing function of z, we have to show that it is an increasing function of c_o.

$$\frac{\partial}{\partial c_o}(u^1 - u^2) = u_o^1 - u_o^2 - \frac{1}{R}(u_1^1 - u_2^2)$$

Here numerical subscripts denote differentiation with respect to the indicated consumption level.

We shall argue that, at the optimum,

$$\frac{1}{R} = \theta\frac{u_o^1}{u_1^1} + (1 - \theta)\frac{u_o^2}{u_2^2} \qquad (19)$$

Given this equation, $\frac{\partial}{\partial c_o}(u^1 - u^2)$ is positive if $\frac{u_o^1}{u_1^1}$ is larger than $\frac{u_o^2}{u_2^2}$ (and conversely). The latter conditions is precisely (ii).

It remains to derive (19) given the assumption that at the optimum we have a moral hazard problem (which follows from (i)). From Figure 4 we know that at D an indifference curve is tangent to the moral hazard constraint. Equating these two slopes we have

$$\frac{\theta(u_o^1 - \frac{1}{R}u_1^1) + (1-\theta)(u_o^2 - \frac{1}{R}u_2^2)}{\theta(1-\theta)(u_1^1 - u_2^2)} = \frac{(u_o^1 - \frac{1}{R}u_1^1) - (u_o^2 - \frac{1}{R}u_2^2)}{(1-\theta)u_1^1 + \theta u_2^2}$$

Crossmultiplying, we obtain (19), which completes the proof.

The first condition in the theorem is the one that we have earlier claimed to be plausible, and which is pretty much required for moral hazard to be present in the optimum. The second condition says that, at equal utilities whether working or not, the individual has a greater incentive to save when not working. Our conclusion is that it is normal for a tax on saving to be associated with the optimal social insurance policy.

Suppose now that saving is allowed, and is not taxed. This means that optimization is further constrained by a saving equilibrium condition, that the derivative of expected utility with respect to c_o is zero.[7] Thus the condition is represented by the locus of points where the indifference curves have vertical slope, the curve FE in Figure 5. As one moves right along this curve, utility, and the size of the social insurance programme, diminish. If individuals made their savings plans under the assumption that they would work in the future, analysis would be completed by combining the savings constraint with the moral hazard constraint $u^1(c_o,c_1) = u^2(c_o,c_2)$ and the optimum would be at E, where the two constraints intersect. However if the individual plans savings and future work at the same time, we have a different condition:

$$\max_s [\theta u^1(c_o - s, c_1 + s) + (1-\theta)u^3(c_o - s, c_2 + s)]$$

$$\geq \max_s [\theta u^2(c_o - s, c_2 + s) + (1-\theta)u^3(c_o - s, c_2 + s)]$$

when savings are optimal assuming work,
This is a more stringent condition/since the ability to adjust savings increases the utility available to those planning not to work. This is shown in Figure 5 as FG with the optimum occurring at H. (This model is explored in more detail in another paper.) Notice that, when individuals are free to save without constraint or taxation, it is still optimal to be on the boundary of the feasible set.[8]

One aspect of these solutions should be further noted. Since they are on the boundary of the feasible set, individuals are in fact indifferent whether to work in period 2 or not. Similarly in the one-period model they are indifferent between working and not working. If this indifference were reflected in random choice rather than an implausible adherence to government wishes, there would, strictly speaking, be no optimum. though the government could adopt policies arbitrarily close to the ones we have discussed under which individuals would certainly wish to work if they could. That is why it is not inappropriate to assume, as we have done, that the government can choose when the consumer is indifferent. Of course a fully satisfactory treatment of these matters would have to model the costs of implementing choices explicitly, and that would take us too far afield.

The chief conclusion of this section is that a government concerned about social insurance would normally want to discourage private saving, for example by taxing it; and that, if it could not do so, the extent of the social insurance programme would be diminished. This exemplifies a general feature of moral hazard situations, that the provider of insurance would usually want to control trade in related commodities. For example, fire insurance

companies should want their clients to buy fire extinguishers; and they like to see high taxes on the consumption of tobacco.

5. The Continuous-Time Model

We turn to a model in which the retirement date is continuously variable, while maintaining the assumption that labour choice at each moment is discrete. This model allows us to ask two further questions: how consumption should be made to vary with age when working, and how the retirement benefit should depend upon the age of retirement. Since the moral hazard constraints can no longer be pictured in a diagram, the mathematics of the model is more complicated than for the previous models, but we shall use the earlier analysis to guide us to the solution.

We make the following assumptions. If everyone were working, output per unit period would be unity. The real interest rate is zero. Utility is taken to be the undiscounted integral of instantaneous utility. When the individual is working, his utility is $u_1(c)$, when not working, his utility is $u_2(c)$ or $u_3(c)$ according to whether he is able to work or not. **All three functions are strictly concave.** To simplify the analysis, we assume from the outset that u_2 and u_3 differ by a constant:

$$u_2(c) = u_3(c) + b, \qquad b \geq 0.$$

Saving is controlled by the government. Consequently, consumption when working can be a function of age, $c_1(t)$; and consumption when retired can be a function both of age t and of age at retirement, r, $c_2(t,r)$. The date s at which an individual becomes unable to work is a random variable, with density function f

and distribution function F. The length of life is denoted by T, and $f(s) > 0$ for $0 < s < T$. There are no atoms in the distribution.

If the age of retirement is r and the age of disability is s, utility is

$$\int_0^r u_1(c_1(t))dt + \int_r^s u_2(c_2(t,r))dt + \int_s^T u_3(c_2(t,r))dt$$

$$= \int_0^r u_1(t)dt + \int_r^T u_2(t,r)dt - b(T - s), \qquad (20)$$

where we have used (19), and introduced the notation

$$u_1(t) = u_1(c_1(t)), \qquad u_2(t,r) = u_2(c_2(t,r))$$

Neither individual nor government can affect the last part of (20). What we are interested in is

$$v(r) = \int_0^r u_1(t)dt + \int_r^T u_2(t,r)dt \qquad (21)$$

and expected utility can be taken to be

$$V(c_1,c_2,r) = \int_0^r v(s)f(s)ds + v(r)[1 - F(r)] \qquad (22)$$

since the individual has to retire at the date of disability s if he has not already done so. In (22), r is the age at which the individual decides to retire if he is still able to work at that date.

(22) expresses the maximand for our problem as an average of v, the expected value of $v(\text{Min}(s,r))$. This average occurs frequently in our analysis, and it is convenient to have a special

notation for it. For any integrable function h, we define

$$J_r(h) = \int_0^r h(s)f(s)ds + h(r)[1 - F(r)] \qquad (23)$$

For later reference we note the following

Properties of J_r

(i) J_r is a positive linear functional for each r.

(ii) $J_r(h)$ is monotonically increasing (or decreasing) in r if h is monotonically increasing (or decreasing).

(iii) $J_r(h)$ is a constant function of r if and only if h is a constant.

(iv) If h is differentiable,

$$\frac{d}{dr}J_r(h) = h'(r)[1 - F(r)] \qquad (24)$$

Proofs. (i) and (iv) follow immediately from (23). To prove (ii), we let $r' > r$, and calculate from (23) that

$$J_{r'}(h) - J_r(h) = \int_r^{r'} [h(s) - h(r)]f(s)ds + [h(r') - h(r)]\int_{r'}^T f(s)ds \qquad (25)$$

Thus $J_{r'}(h) > J_r(h)$ if h is monotonically increasing. The case where h is monotonically decreasing is proved similarly.

From (25) it also follows that $J_r(h)$ is constant if h is constant. To complete the proof of (iii), suppose that $J_r(h) = c$, a constant, for all r. From (23) it follows that $h(r)$ is differentiable, and (24) then implies $h' = 0$, i.e. h is a constant.

The net resource cost of an individual who retires at r and becomes unable to work at s is

$$z(r) = \int_0^r c_1(t)dt + \int_r^T c_2(t,r)dt - r \qquad (26)$$

recalling that a worker produces unit output. If the government requires A units of output for its own purposes, the aggregate resource constraint is

$$J_r(z) + A \leq 0 \qquad (27)$$

We can now state formally the problem to be addressed. We are to

Maximize $J_r(v)$ subject to (27) and

$$J_t(v) \leq J_r(v), \quad \text{all } t \leq r \qquad (28)$$

This last constraint says that individuals do not prefer to retire before r. Since the government can set consumption levels equal to zero for anyone who retires after a date it chooses, we need not be concerned about individual plans to work too long. Thus the effective constraint on government is that it choose r, and the functions c_1 and c_2 so that (27) and (28) hold.

In the full, first-best optimum, the government is constrained only by (27), and marginal utilities of consumption should be equated in all circumstances:

$$c_1(t) = c_1^o, \quad c_2(t,r) = c_2^o; \quad \text{with } u_1'(c_1^o) = u_2'(c_2^o) \qquad (29)$$

If under these circumstances we had $u_1(c_1^o) \geq u_2(c_2^o)$, there would

be no moral hazard problem, for, as one can see from (21), $v(r)$ would be a nondecreasing function of r, and $J_t(v)$ consequently a nondecreasing function of t.

In the case where there is no moral hazard problem, the optimal value of r is normally T. To see this consider the derivatives of $J_r(v)$ and $J_r(z)$ with respect to r. Using property (iv) of J_r, we have

$$\frac{\partial}{\partial r}J_r(v) = \{u_1(c_1^o) - u_2(c_2^o)\}\{1 - F(r)\}$$

$$\frac{\partial}{\partial r}J_r(z) = \{c_1^o - c_2^o - 1\}\{1 - F(r)\}$$

Thus $\frac{\partial}{\partial r}J_r(v) > 0$ unless $r = T$, except in the borderline case with $u_1(c_1^o) = u_2(c_2^o)$; and $\frac{\partial}{\partial r}J_r(z) \leq 0$ normally, e.g. when $u_1' = u_2'$ implies $c_1^o \leq c_2^o$, or if $A \geq 0$, for $-A = J_r(z) = c_2^o T + (c_1^o - c_2^o - 1)\{\int_o^r sf ds + r[1 - F(r)]\}$. Thus it cannot in these cases be optimal to have $r < T$, since an increase in r would increase utility without increasing the resource cost.

The interesting case is where $u_1' = u_2'$ is inconsistent with $u_1 \geq u_2$. Accordingly we assume from now on that

$$u_1(c_1) = u_2(c_2) \quad \text{implies} \quad u_1'(c_1) \leq u_2'(c_2) \tag{30}$$

Before proceeding with the analysis, we note a convenient feature of the optimum. Once a man has retired, there is no advantage from inefficient intertemporal allocation of his consumption. Incentives to work depend on expected utility as a function of the retirement date, so the cost of providing expected utility conditional on retirement should be minimized. Thus

$$c_2(t,r) = c_2(r) \tag{31}$$

This means that v and z can be written

$$v(r) = \int_0^r u_1(t)dt + u_2(r)(T - r) \tag{32}$$

$$z(r) = \int_0^r c_1(t)dt + c_2(r)(T - r) - r \tag{33}$$

The bad feature of the maximization problem we have to deal with is a lack of the concavity conditions that would render necessary conditions for the optimum also sufficient. The moral hazard constraints (28) cannot be expected to be concave in any of the control variables c_1, c_2, or r. But if we regard u_1 and u_2 as the control variables, the maximand, and the constraints (28), are linear, and the resource constraint (27) is convex in them. Thus we have a well-behaved programming problem if r is taken as given. We shall first derive sufficient conditions for the optimum conditional upon a specified value of r; and afterwards derive a necessary condition for optimality with respect to r.

The earlier models yielded solutions in which the individual was indifferent whether to work or not. Correspondingly, one can expect that in the present model, the individual will be indifferent about the age of retirement. We first consider the best path with this property and then show that it is indeed optimal. If $J_t(v)$ is a constant independent of t, then $v(r)$ is a constant independent of r, by property (iii). We write

$$\int_0^r u_1(t)dt + u_2(r)(T - r) = \bar{v} \tag{34}$$

Given that v is constant, we ask what is the optimal combination

of u_1 and u_2. It is the one that for the given value of $J_r(z)$ maximizes $\bar{v}$, or equivalently for given $\bar{v}$ minimizes $J_r(z)$.

Let G_1 and G_2 be the inverse functions of $u_1(c)$ and $u_2(c)$, so that $c_1(t) = G_1(u_1(t))$ and $c_2(t) = G_2(u_2(t))$. Then

$$z(r) = \int_0^r G_1(u_1)dt + G_2\left(\frac{1}{T-r}[\bar{v} - \int_0^r u_1 dt]\right)(T-r) - r \qquad (35)$$

and it follows that

$$\frac{\partial z(r)}{\partial u_1(t)} = \begin{cases} G_1'(u_1(t)) - G_2'(u_2(r)) & (t \leq r) \\ 0 & (t > r) \end{cases} \qquad (36)$$

Consequently for $t \leq r$

$$\frac{\partial}{\partial u_1(t)} J_r(z) = \int_t^r [g_1(t) - g_2(s)]f(s)ds + [g_1(t) - g_2(r)][1 - F(r)], \qquad (37)$$

where we have introduced the notations

$$\left.\begin{aligned} g_1(t) &= G_1'(u_1(t)) = 1/u_1'(c_1(t)) \\ g_2(t) &= G_2'(u_2(t)) = 1/u_2'(c_2(t)) \end{aligned}\right\} \qquad (38)$$

For cost-minimization, the derivative (37) should vanish for all t:

$$\int_t^r [g_1(t) - g_2(s)]f(s)ds + [g_1(t) - g_2(r)][1 - F(r)] = 0. \qquad (39)$$

Differentiating with respect to t, and using a $\cdot$ for a time derivative we obtain

$$[1 - F(t)]\dot{g}_1(t) = [g_1(t) - g_2(t)]f(t) \qquad (40)$$

which can be written equivalently as

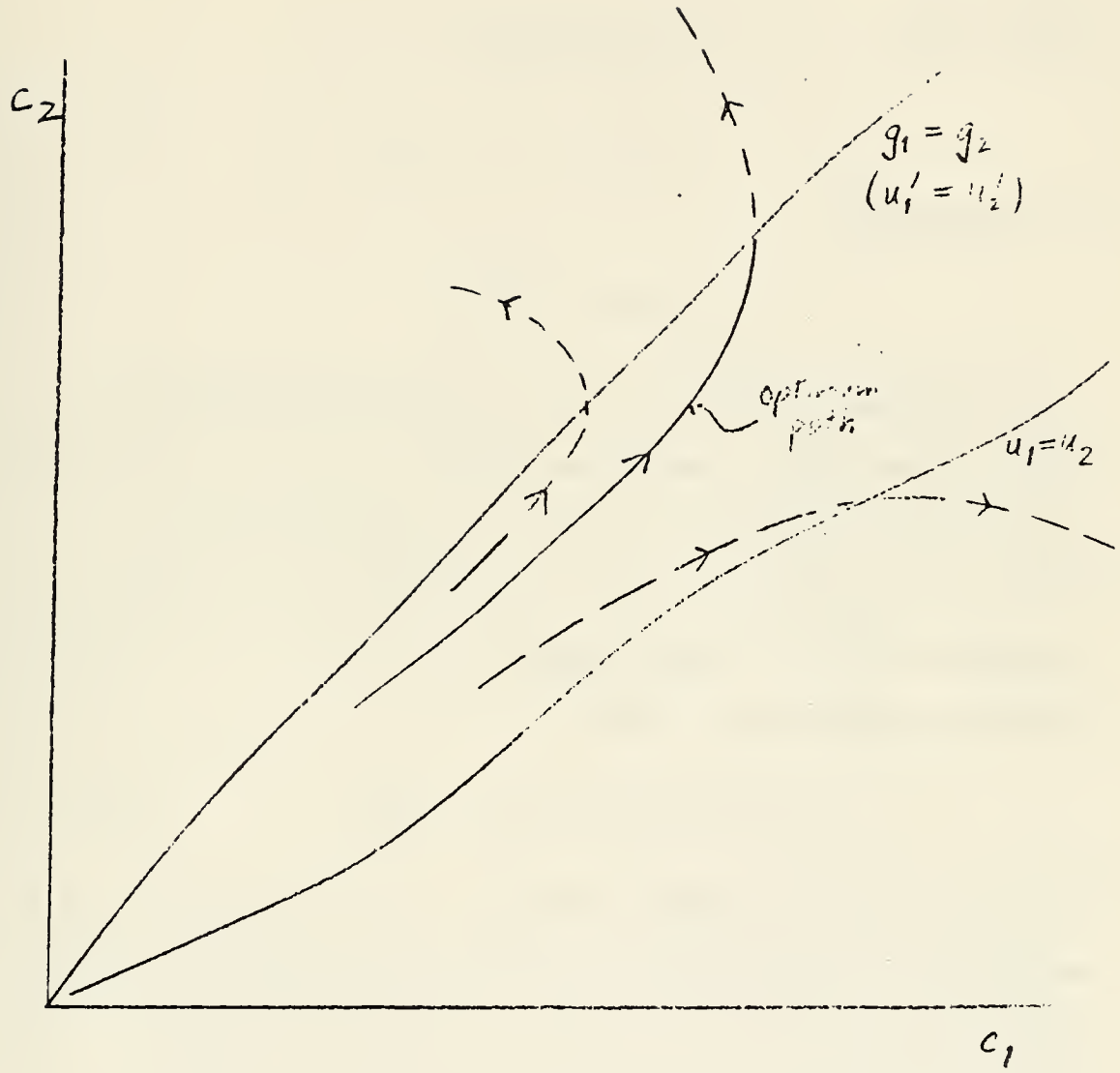

Figure 6

$$[1 - F(t)]\frac{d}{dt}\frac{1}{u_1'(c_1(t))} = \left[\frac{1}{u_1'(c_1(t))} - \frac{1}{u_2'(c_2(t))}\right]f(t)$$

Also, setting $t = r$ in (39), we have

$$g_1(r) = g_2(r) \tag{41}$$

or, equivalently,

$$u_1'(c_1(r)) = u_2'(c_2(r))$$

We have shown that if it is true that there is indifference about retirement age at the optimum, then v is constant, and (40) and (41) hold. We shall show that this is indeed the optimum, and to do this we need to know a little more about the proposed solution. It satisfies two differential equations, (40), and one we obtain by differentiating (34) with respect to r:

$$(T - t)\dot{u}_2(t) = u_2(t) - u_1(t) \tag{42}$$

In Figure 6, we have a phase diagram for this pair of differential equations, in (c_1,c_2)-space. u_1 and u_2 are of course increasing functions of c_1 and c_2. g_1 and g_2 are also increasing functions of c_1 and c_2 respectively; since g_1 is equal to $1/u_1'(c_1)$.

From (40), c_1 is stationary when $g_1 = g_2$; while from (42) c_2 is stationary when $u_1 = u_2$. By assumption (30), the curve $u_1 = u_2$ lies below the curve $g_1 = g_2$ (i.e. $u_1' = u_2'$). Between the two curves, both c_1 and c_2 are increasing with t. (41) says that the solution we propose has to hit the $g_1 = g_2$ curve at $t = r$. Therefore both c_1 and c_2 are increasing functions of t throughout the solution.

Theorem 3 Assume that $u_1 = u_2$ implies $u_1' \leq u_2'$. If

$$\frac{du_2}{dt} = \frac{u_2 - u_1}{T - t}$$

$$\frac{dg_1}{dt} = (g_1 - g_2)\frac{f}{1 - F}$$

$$g_1 = g_2 \quad \text{at} \quad r$$

and $J_r(z) + A = 0$, the policies so defined are optimal for given r. If the optimum for a given r exists, it is unique.

Assuming that $u_1(0) = u_2(0) = -\infty$, a solution with the stated properties exists, for any r between 0 and T such that

$$\int_0^r [1 - F(s)]ds > A \tag{43}$$

(i.e. such that positive consumption is feasible).

We first proved this theorem by introducing Lagrange multipliers for the constraints, and following the standard method of proving sufficiency of the first-order conditions for constrained maximization problems. This method is rather involved. The proof we give here, though less clearly motivated in that Lagrange multipliers are not introduced explicitly, is briefer. We begin by establishing some preliminary results. The policies satisfying the conditions of the theorem are denoted by asterisks. The outcome is compared with alternative policies satisfying the constraints of the maximization problem.

Lemma 1 $J_r(g_2^* v) \geq g_1^*(0) J_r(v)$ for any v satisfying $J_t(v) \leq J_r(v)$.

Since c_1 is an increasing function of t, g_1^* is an increasing function of t, and we have

$$\int_0^r \dot{g}_1^*(t)[J_r(v) - J_t(v)]dt \geq 0 \tag{44}$$

Now

$$\int_0^r \dot{g}_1^* J_r(v)dt = [g_1^*(r) - g_1^*(0)]J_r(v) \tag{45}$$

and

$$\int_0^r \dot{g}_1^* J_t(v)dt = \int_0^r \dot{g}_1^*(t) \int_0^t v(s)f(s)dsdt + \int_0^r \dot{g}_1^*(t)v(t)[1 - F(t)]dt$$

$$= \int_0^r [g_1^*(r) - g_1^*(t)]v(t)f(t)dt + \int_0^r [g_1^*(t) - g_2^*(t)]v(t)f(t)dt$$

by reversing the order of integration in the first integral, and using (40) in the second integral. This last expression simplifies to

$$\int_0^r \dot{g}_1^* J_t dt = g_1^*(r) \int_0^r vfdt - \int_0^r g_2^* vfdt = g_1^*(r)J_r(v) - J_r(g_2^* v) \tag{46}$$

since $g_1^*(r) = g_2^*(r)$.

Combining (44), (45), and (46) we obtain Lemma 1.

Lemma 2 $\quad J_r(g_2^*) = g_1^*(0)$

The second and third conditions of the theorem are equivalent to the vanishing of (37) for all t. Putting $t = 0$, we obtain the stated result.

Proof of the Theorem

We now show that any feasible path does not have higher expected utility.

From the definition of z we have

$$z(s) - z^*(s) = \int_0^s [G_1(u_1) - G_1(u_1^*)]dt + [G_2(u_2(s)) - G_2(u_2^*(s))](T - s)$$

Since G_1 and G_2 are convex functions of their arguments, we can use the inequality for convex functions to obtain

$$z(s) - z^*(s) \geq \int_0^s g_1^*(t)(u_1(t) - u_1^*(t))dt + g_2^*(s)(u_2(s) - u_2^*(s))(T - s)$$

Using the definition of v, (32), and the constancy of $v^*(s)$, we can write this as

$$z(s) - z^*(s) \geq \int_0^s (g_1^*(t) - g_2^*(s))(u_1(t) - u_1^*(t))dt + g_2^*(s)(v(s) - \bar{v}^*) \qquad (47)$$

$$= \int_0^T \frac{\partial z^*(s)}{\partial u_1^*(t)}(u_1(t) - u_1^*(t))dt + g_2^*(s)(v(s) - \bar{v}^*)$$

The second step follows from the expression for the derivative, (36). We now wish to apply the linear operator J_r to (47). The first term on the right hand side vanishes since (37) is zero for all t

$$J_r\left(\int_0^T \frac{\partial z^*(r)}{\partial u_1^*(t)}(u_1(t) - u_1^*(t))dt\right) = \int_0^T J_r\left(\frac{\partial z^*(r)}{\partial u_1^*(t)}\right)(u_1(t) - u_1^*(t))dt = 0.$$

Thus we have

$$J_r(z) - J_r(z^*) \geq J_r(g_2^* v) - J_r(g_2^*)\bar{v}$$

$$\geq g_1^*(0)[J_r(v) - \bar{v}] \qquad (48)$$

by Lemmas 1 and 2.

Now $J_r(z) + A \leq 0$, and $J_r(z^*) + A = 0$. Therefore (48) implies that

$$J_r(v) \leq \bar{v},$$

i.e., expected utility is no greater on the alternative path than on the one satisfying the conditions of the theorem. This proves the sufficiency part of the theorem.

We next prove the existence part of the theorem. Refer to the diagram. Since marginal utilities go to infinity as consumption goes to zero, g_1 and g_2 then go to zero, and the $g_1 = g_2$ curve passes through the origin. Similarly the curve $u_1 = u_2$ passes through the origin.

The value of $g_1(r)$ determines the point on the curve $g_1 = g_2$ at which the solution path ends, and therefore the whole path. By choosing $g_1(r)$ small enough, aggregate consumption, $z(t) + t$, can be made as small as we please, uniformly for $0 \leq t \leq r$. Thus $J_r(z(t) + t)$ can be made as small as we please.
$J_r(t) = \int_0^r tf(t)dt + r - rF(r) = \int_0^r [1 - F(t)]dt$ on integrating by parts. Therefore if (43) holds, $J_r(z) + A$ can be made negative by choosing $g_1(r)$ small enough.

We now have to show that $J_r(z) + A$ can be made positive by choosing $g_1(r)$ large enough. By equation (40), we have

$$\frac{d}{dt}\log g_1 = \left(\frac{g_2}{g_1} - 1\right)\frac{d}{dt}\log\,[1 - F(t)]$$

$$\leq -\frac{d}{dt}\log\,[1 - F(t)],$$

since $g_1/g_2 \geq 0$ and $1 - F$ decreases with t. Integrating from 0 to r, we obtain

$$g_1(0) \geq g_1(r)[1 - F(r)] \qquad (49)$$

Therefore by choosing $g_1(r)$ large enough, $c_1(0)$, and all $c_1(t)$ for $0 \leq t \leq r$, can be made as large as we wish, thus increasing $J_r(z)$ without limit.

The proof of existence is complete.

The rather restrictive condition about utility functions introduced to prove existence is by no means necessary. As will be evident from the proof, the condition was introduced merely to exclude cases with zero consumption, and their attendant corner conditions.

Throughout the argument, r has been taken to be less than T. The form of the differential equations makes it clear that the case $r = T$ would need careful handling; and the above existence proof would not go through at all. Fortunately, it turns out that, in the most plausible cases, the optimal value of r is less than T. This will emerge as a by-product of our discussion of the optimization of r.[9]

We have shown that, for a given value of A, and for a given $r < T$, an optimum exists. The optimum is given by a solution of our differential equations which at time r takes values $c_1(r)$, $c_2(r)$ satisfying $u_1'(c_1(r)) = u_2'(c_2(r))$. Denote utility arising from the

the solution to (40)-(42) $c_1(r) = x$ by $V(r,x)$, and denote the value of $J_r(z)$ by $Z(r,x)$. We first note, without taking the space for formal proof, that V and Z are differentiable functions of r and x.

If, for given r, x satisfies $Z(r,x) + A = 0$, it follows from Theorem 3 that the solution path so defined yields an optimum. If utility is strictly concave, the optimum for given r is unique. Therefore the equation $Z(r,x) + A = 0$ has a unique solution, and we can define a function $x(r)$ by the equation

$$Z(r,x(r)) + A = 0 \tag{50}$$

for the set of r which are feasible,

Lemma 3 / $x(r)$ is a continuous function for $r < T$.

To prove this, we must establish that when r is restricted by $r \leq \bar{r} < T$, (50) implies that $x(r)$ is bounded. From the differential equations we have the inequalities

$$\frac{dg_1}{dt} < g_1 \frac{f}{1 - F}, \qquad \frac{du_2}{dt} < \frac{u_2}{T - t}$$

Integrating from 0 to r, we deduce that

$$\begin{aligned} g_1(0) &> g_1(r)[1 - F(r)], & u_2(0) &> u_2(r)(T - r) \\ &\geq g_1(r)[1 - F(\bar{r})] & &\geq u_2(r)(T - \bar{r}) \end{aligned} \tag{51}$$

Also $c_1(0) \leq c_1(t) \leq c_1(r)$, $c_2(0) \leq c_2(t) \leq c_2(r)$; and $c_2(r)$ and $c_1(r) = x$ are related by $u_1' = u_2'$. Therefore we have upper and lower bounds on all $c_1(t)$, which are increasing functions of x. It follows that equation (50) imposes an upper bound on x. Now consider a convergent sequence $r_\nu \to r_o < \bar{r} (< T)$. Since $r_\nu \leq \bar{r}$

eventually, $\{x(r_\nu)\}$ is bounded, and has a limit point, x_o. Z being continuous, we have $Z(r_o,x_o) + A = 0$. Thus x_o is unique and equal to $x(r_o)$. This proves the lemma.

The lemma implies further that $V(r) = V(r,x(r))$ is a continuous function of r, and that, as r varies continuously, optimal $c_i(t)$ all vary continuously. We next establish that $V(r)$ is a differentiable function of r and obtain a formula (66) for its derivative. The argument used is similar to that employed in Theorem 3, where we expressed $z(s)$ in terms of the functions u_1 and v. In the present case, we must identify dependence on the retirement date r, and this will be done by superfixes. We have

$$z^r(s) = \int_o^s c_1^r(t)dt + c_2^r(s)(T - s) - s \qquad (52)$$

with

$$c_1^r(t) = G_1(u_1^r(t))$$

$$c_2^r(s) = G_2\left(\frac{1}{T - s}\{V(r) - \int_o^s u_1^r(t)dt\}\right).$$

Since the resource constraint is fixed,

$$J_{r'}(z^{r'}) - J_r(z^r) = 0 \qquad (53)$$

for any r, $r' < T$. We shall manipulate the left hand side of (53) and eventually let $r' \to r$.

Since the function u_1^r is so chosen as to minimize $J_r(z^r)$,

$$J_r(z^r) \leq J_r(\hat{z}) \qquad (54)$$

where

$$\hat{z}(s) = \int_0^s c_1^{r'}(t)dt + G_2\left(\frac{1}{T-s}\{V(r) - \int_0^s u_1^{r'}(t)dt\}\right)(T-s) + s.$$

By the mean value theorem, we have

$$\hat{z}(s) - z^{r'}(s) = G_2'\left(\frac{1}{T-s}\{V_o(r,r',s) - \int_0^s u_1^{r'}(t)dt\}\right)\{V(r) - V(r')\} \quad (55)$$

where $V_o(r,r',s)$ lies between $V(r)$ and $V(r')$. G_2' here tends to $g_2^{r'}(s)$ as $r \to r'$; and to $g_2^r(s)$ as $r' \to r$. Apply the operator J_r to (55), and use (53), (54):

$$J_{r'}(z^{r'}) - J_r(z^{r'}) \leq \{J_r(g_2^r) + \varepsilon_1(r,r')\}\{V(r) - V(r')\} \quad (56)$$

where $\varepsilon_1(r,r') \to 0$ as $r' \to r$.

Reversing the roles of r and r', and using the fact that $J_r(g_2^r)$ is continuous in r, we also have

$$J_{r'}(z^r) - J_r(z^r) \geq \{J_r(g_2^r) + \varepsilon_2(r,r')\}\{V(r) - V(r')\} \quad (57)$$

where $\varepsilon_2(r,r') \to 0$ as $r' \to r$.

Applying the mean value theorem to the left hand side of (56), recalling property (iv) of J_r, that $\frac{\partial}{\partial r}J_r(h) = h'(r)[1 - F(r)]$, and using the fact, derived from (42) and (52), that

$$\frac{\partial}{\partial s}z^r(s) = c_1^r(s) - c_2^r(s) + g_2^r(s)[u_2^r(s) - u_1^r(s)] - 1 \quad (58)$$

$$= \frac{H^r(s)}{1-F(r)}, \quad \text{say,}$$

we obtain

$$J_{r'}(z^{r'}) - J_r(z^{r'}) = H^{r'}(\hat{r})(r' - r), \quad (59)$$

with $\hat{r}$ between r and r'. By the continuity of consumption levels both with respect to time and with respect to the terminal date, we see that $H^r(\hat{r}) \to H^r(r)$ as $r' \to r$. Therefore (56) and (59) imply that

$$\{H^r(r) + \eta_1(r,r')\}(r' - r) \leq \{J_r(g_2^r) + \varepsilon_1(r,r')\}\{V(r) - V(r')\} \quad (60)$$

where both ε_1 and η_1 tend to zero as $r' \to r$.

Reversing the roles of r and r', and using the continuity of $H^r(r)$ and $J_r(g_2^r)$ with respect to r, we also obtain from (60),

$$\{H^r(r) + \eta_2(r,r')\}(r' - r) \geq \{J_r(g_2^r) + \varepsilon_2(r,r')\}\{V(r) - V(r')\} \quad (61)$$

where ε_2 and η_2 tend to zero as $r' \to r$. Now consider $r' > r$. (60) and (61) imply

$$\frac{H^r(r) + \eta_1}{J_r(g_2^r) + \varepsilon_1} \leq -\frac{V(r') - V(r)}{r' - r} \leq \frac{H^r(r) + \eta_2}{J_r(g_2^r) + \varepsilon_2} \quad (62)$$

since $J_r(g_2^r) > 0$. Letting $r' \to r$, (62) implies that the limit of $\{V(r') - V(r)\}/(r' - r)$ as r' tends to r from above exists, and is equal to $H^r(r)/J_r(g_2^r)$. A similar argument with $r' < r$ establishes that $\{V(r') - V(r)\}/(r' - r)$ tends to the same limit as r' tends to r from below. We have therefore established that $V(r)$ is differentiable, and that

$$V'(r) = -\frac{H^r(r)}{J_r(g_2^r)} \quad (63)$$

We now express $H^r(r)$, defined by (58), in a slightly different way. Recollect that $g_1^r(r) = g_2^r(r)$. Then we can write

$$H^r(r) = \{c_1^r(r) - g_1^r(r)u_1^r(r) - c_2^r(r) + g_2^r(r)u_2^r(r) - 1\}[1 - F(r)]$$

$$= \{- h_1(r) + h_2(r) - 1\}[1 - F(r)] \qquad (64)$$

where we define

$$h_i(r) = g_i^r(r)u_i^r(r) - c_i^r(r), \qquad r = 1,2$$

$$= \frac{u_i(c_i^r)}{u_i'(c_i^r)} - c_i^r \text{ evaluated at } r \qquad (65)$$

Thus we have, finally, the formula

$$V'(r) = \frac{1 - F(r)}{J_r(g_2^r)}[h_1(r) - h_2(r) + 1] \qquad (66)$$

We must now consider the sign of $h_1 - h_2 + 1$ along the curve $g_1 = g_2$, since that curve contains the locus of end-points for r-optimal paths as r varies. A straightforward calculation yields

$$\frac{dh_i}{dg_i} = u_i \qquad (67)$$

From this it follows that, as $g = g_1 = g_2$ varies,

$$\frac{d}{dg}(h_1 - h_2) = u_1 - u_2$$

$$< 0, \qquad (68)$$

since $u_1 < u_2$ on the curve $g_1 = g_2$.

The inequality (68) implies that there is at most one point $(\bar{c}_1, \bar{c}_2)$ on the curve $g_1 = g_2$ at which $h_1 - h_2 + 1 = 0$, and at which therefore $V'(r) = 0$ if $c_i^r(r) = \bar{c}_i$, $i = 1,2$. If $(c_1^r(r), c_2^r(r))$ lies to the left of that point $V'(r) > 0$; if it lies

to the right $V'(r) < 0$. If there exists such a point, and if there exists $r^* < T$ for which $c_i^r(r^*) = \bar{c}_i$, $i = 1,2$, this r^* defines the optimal policy. We have not argued that there is at most one value of r satisfying $C_i^r(r) = \bar{c}_i$.

Lemma 4 If for $i = 1,2$, $u_i'(0) = \infty$, $u_i'(\infty) = 0$; and if there exists $a > 0$ such that

$$u_2 - u_1 \geq a \quad \text{when} \quad u_1' = u_2' \tag{69}$$

then there exists $\bar{c}_1$, $\bar{c}_2$ such that

$$h_1(\bar{c}_1) - h_2(\bar{c}_2) = -1, \qquad u_1'(\bar{c}_1) = u_2'(\bar{c}_2) \tag{70}$$

Proof. Consider first how $h_i(c_i)$ behaves as $c_i \to 0$, or, equivalently, $g \to 0$. Let ε be an arbitrarily small positive number. Since u_i is concave,

$$u_i(c_i) + u_i'(c_i)(\varepsilon - c_i) \geq u_i(\varepsilon)$$

Therefore

$$h_i(c_i) = \frac{u_i(c_i)}{u_i'(c_i)} - c_i \geq \frac{u_i(\varepsilon)}{u_i'(c_i)} - \varepsilon$$

$$\to -\varepsilon \quad \text{as} \quad c_i \to 0$$

It follows that if u_i is negative for small c_i, $\operatorname{Lim} h_i(c_i) = 0$; while if $u_i(0) \geq 0$, it is also true that $\operatorname{Lim} h_i(c_i) = 0$. Thus

$$\operatorname*{Lim}_{g \to o} [h_1(c_1) - h_2(c_2)] = 0 \tag{71}$$

Now consider what happens as $c_1, c_2 \to \infty$; or, equivalently, since $\operatorname{Lim} u_i' = 0$, $g \to \infty$. From (68), (69), and the assumption that

$g \to \infty$, $h_1 - h_2$ decreases without limit:

$$\lim_{g \to \infty} [h_1(c_1) - h_2(c_2)] = -\infty \tag{72}$$

Since $h_1 - h_2$ is a continuous decreasing function of g, and (71), (72), hold, a solution of (70) must exist, and the lemma is proved.

Lemma 5 Let

$$\phi(t) = \frac{(T - t)f(t)}{1 - F(t)} \tag{73}$$

be bounded as $t \to T$. If $x(r)$ is defined, by

$$Z(r, x(r)) + A = 0,$$

and $A < \int_0^T [1 - F(t)]dt$ (so that positive consumption is feasible), then

$$x(r) \to \infty \quad \text{as} \quad r \to T \tag{74}$$

Proof Let M be a positive number, and consider paths optimal for some A with

$$c_1(r) \leq M \tag{75}$$

On any optimum path, $g_1 > g_2$, $u_1 < u_2$; and there exists a positive number α, dependent on M but independent of r, such that for all of these paths and for all t, $0 \leq t \leq r$,

$$\text{Max } \{g_1 - g_2, \; u_2 - u_1\} \geq \alpha \tag{76}$$

If this were not so, then by compactness and continuity, we should have $g_1 \leq g_2$ and $u_2 \leq u_1$, for some r and t, which is impossible by assumption.

From (76) and the differential equations satisfied by the

optimal path, it follows that

$$\frac{d}{dt}(g_1 + u_2) = (g_1 - g_2)\frac{f}{1 - F} + (u_2 - u_1)\frac{1}{T - t}$$

$$\geq \alpha \text{ Min}\left(\frac{f}{1 - F}, \frac{1}{T - t}\right) \qquad (77)$$

Since ϕ defined in (73) is bounded, there exists $N > 1$ such that

$$\frac{(T - t)f(t)}{1 - F(t)} \leq N,$$

and therefore (77) implies

$$\frac{d}{dt}(g_1 + u_2) \geq \frac{\alpha}{N}\frac{f}{1 - F}$$

$$= -\frac{\alpha}{N}\frac{d}{dt}\log\,[1 - F(t)]$$

Integrating from t to r we obtain

$$g_1(t) + u_2(t) \leq g_1(r) + u_2(r) + \frac{\alpha}{N}\log\,[1 - F(r)] - \frac{\alpha}{N}\log\,[1 - F(t)]$$

Therefore if $r \to T$ and the corresponding r-optimal paths all satisfy (75), $g_1^r(t) + u_2^r(t) \to -\infty$ for any fixed $t < T$. Thus $c_1^r(t)$, $c_2^r(t) \to 0$ as $r \to T$, and $J_r(z) \to -\int_0^T [1 - F(t)]dt$. It follows that for all sufficiently large r, $J_r(z) + A = 0$ is inconsistent with (75). Therefore $x(r) \to \infty$ as asserted.

It should be noted that the assumption that ϕ is bounded is rather a weak one, automatically satisfied if $f(t)$ tends to a non-zero limit as $t \to T$. f would have to be very pathological for the result of Lemma 5 to fail.

When the assumptions of these Lemmas are satisfied, it follows that the optimal r^* is less than T. For when $x(r) > \bar{c}_1$,

$V'(r) < 0$, and this must be true for all sufficiently large r. Since V is evidently continuous at T, T cannot be the optimal value of r.

Summarizing the above analysis, we have

Theorem 4 Under the assumptions of Theorem 3, assumption (69), and the assumption that marginal utilities tend to zero as consumption levels tend to infinity, there exists an optimal $r < T$. The optimal path or paths is identified by the unique $(\bar{c}_1, \bar{c}_2)$ satisfying the additional condition

$$h_2(r) - h_1(r) = 1 \tag{78}$$

where h_1, h_2 are defined in (65).

It should be recognized that the condition (69) used to prove that optimal r is less than T is excessively strong, though quite plausible. It states that the marginal disutility of labour is bounded away from zero for large consumption. In cases where T is the optimal r, it will be optimal to have c_1 and c_2 tend to infinity as t tends to T. Such cases are therefore rather peculiar.

6. Features of Solutions

Several aspects of the optimum emerge from the general analysis and call for explicit notice. In the first place, since v is constant, the consumer is indifferent about the date of his retirement. In effect, he is assumed to retire when the government wishes him to. A small deviation from the optimum would make the consumer strictly prefer r to any other retirement date. But the government wishes

consumers to retire before the end of life (and could set consumption for later retirers to zero to achieve this). This latter feature contrasts with the situation where the full optimum is achievable in which, normally, everyone who can work does. Surprisingly, that curious feature of the real world, the compulsory retirement age, may have some optimality properties.

Secondly, since the lifetime utility of a man who loses his ability at s is $v(s) - (T - s)b$, the entire cost of ill-health is borne by the sufferer,[10] but this is the only source of difference among the utilities of different individuals. If b were zero, everyone would have the same lifetime utility, while the marginal utility of income consumption in different periods would vary from individual to individual. Insurance would be perfect in a naive sense, not the economist's sense.

Thirdly, it may seem curious that the solution is independent of b, that is, independent of the effect of ill-health. The reason is that marginal utilities are, by assumption, unaffected by the state of health of the consumer. In the more general case where the difference between u_2 and u_3 varies with c_2, the manner of dependence does affect the optimum.

Fourthly, it is worth summarizing the basic features of the optimum that emerge from the diagram:

$$u_2(c_2) > u_1(c_1) \tag{79}$$

Thus an individual would always be immediately better off if he retired. He is prepared to continue working because of improved retirement benefits with additional work. It may be helpful to consider why one would not want $u_2 = u_1$. Starting from such a

situation, an increase in c_2 at retirement date, financed by a decrease in c_1 earlier, increases expected utility from planning retirement at that date, since it transfers consumption to the state with higher marginal utility. Thus work is not discouraged, and utility increased. The argument works at r, and therefore also at earlier dates.[11]

We also have

$$u_2'(c_2) > u_1'(c_1) \tag{80}$$

Thus the individual would prefer more insurance. Moral hazard keeps the extent of social insurance down.

We recall also the basic intertemporal facts, that c_1 and c_2 are differentiable functions of time, and that

$$\frac{dc_1}{dt} > 0 \tag{81}$$

$$\frac{dc_2}{dt} > 0 \tag{82}$$

There is no difficulty in principle about calculating optimal policies. First the terminal point of the optimal path has to be found, by solving simultaneously $u_1' = u_2'$ and $h_2 - h_1 = 1$. Paths are calculated backwards from there with various values of r. In each case, $J_r(z)$ is calculated, and we have the solution for that particular value of the resource constraint. We illustrate the procedure with a simple case.

Example

$$u_1 = \log c, \qquad u_2 = a + \log c, \qquad a > 0$$

$$f(s) = 1, \qquad T = 1$$

It is readily checked that Theorems 3 and 4 apply. In terms of c_1 and c_2, the differential equations in Theorem 3 are

$$(1 - t)\frac{\dot{c}_2}{c_2} = \log \frac{c_2}{c_1} + a$$

$$(1 - t)\dot{c}_1 = c_1 - c_2$$

The terminal point is given by $c_1 = c_2$ and

$$c_1(\log c_1 - 1) - c_2(\log c_2 + a - 1) + 1 = 0,$$

so that $c_1 = c_2 = 1/a$.

To solve the differential equations, we define

$$w = \log \frac{c_1}{c_2}, \qquad m = \log (1 - t) - \log (1 - r).$$

In terms of these variables, we have

$$\frac{dw}{dm} = e^{-w} - w + a - 1, \qquad w = 0 \text{ at } m = 0$$

$$\frac{dc_2}{dm} = c_2(w - a), \qquad c_2 = \frac{1}{a} \text{ at } m = 0$$

Solution of these equations gives the optimal path for any value of r. For each r we then compute $J_r(z)$, which is found, after a little manipulation, to be

$$J_r(z) = \int_0^r (1 - t)(c_1 + c_2 - 1)dt + \frac{(1 - r)^2}{a}$$

$$= (1 - r)^2 \left[\int_0^{-\log(1-r)} (c_2 + c_2 e^{-w} - 1)e^{2m} dm + \frac{1}{a} \right]$$

In this case, then, it is relatively easy to obtain the solution for different resource constraints, since only one solution of the differential equations is required.

In table I we give solutions for[12] $a = .5$, and three values of the government net subsidy to the scheme, $-A$, .5, 0, and $-.2$. Table II displays solutions for a case with much greater disutility of work, $a = 1$, and the same values of the net subsidy. It should be noted that in table I, c_1 and c_2 are equal to 2 at r. This is a very high figure relative to the wage, which is unity. A rather implausible assumption about the disutility of work is required to reduce $c_1(r)$ and $c_2(r)$ below unity. But in all cases people receive consumption during most of working life that is less than the wage plus the net subsidy, and only a very small proportion of the population obtain a retirement benefit that is greater than wage plus net subsidy.

Two other features of the numerical results deserve comment. In all cases $c_1 - c_2$ increases with t until t is very close to r.

The second feature is that in table I, r is nearly 1, even when there is a very large government deficit ($A = -.5$). Thus the direct effect of moral hazard, arising from the consumer's ability to retire healthy without special penalty, is almost

Table I

a = .5

t	A = -.5 c_1	A = -.5 c_2	t	A = 0 c_1	A = 0 c_2	t	A = .2 c_1	A = .2 c_2
0	1.00	.77		.50	.38		.30	.23
.1	1.03	.78		.51	.39		.31	.24
.2	1.05	.81		.53	.40		.32	.24
.3	1.09	.83		.54	.42		.33	.25
.4	1.13	.86		.56	.43		.34	.26
.5	1.17	.90		.59	.45		.35	.27
.6	1.24	.95		.62	.47		.37	.28
.7	1.32	1.02		.66	.51		.40	.31
.8	1.45	1.12		.73	.56		.44	.33
.9	1.69	1.34		.86	.66		.51	.39
.95	1.92	1.64		1.01	.77		.60	.46
.99				1.46	1.13		.88	.67
	r = .969			r = .998			r = 1.000 (more accurately $1 - 1.8 \times 10^{-4}$)	

Table II

a = 1

t	A = -.5 c_1	A = -.5 c_2	t	A = 0 c_1	A = 0 c_2	t	A = .2 c_1	A = .2 c_2
0	.87	.59		.49	.28		.30	.17
.1	.90	.63		.52	.30		.31	.18
.2	.93	.69		.54	.32		.33	.19
.3	.96	.75		.57	.34		.35	.20
.4	.99	.85		.61	.36		.37	.21
.5				.66	.39		.40	.23
.6				.72	.44		.44	.25
.7				.80	.51		.50	.29
.8				.91	.65		.59	.35
.9							.78	.49
.95							.96	.76
.99								
	r = .494			r = .882			r = .963	

entirely eliminated: hardly anyone retires early. Nevertheless the outcome is far from the first-best optimum that would be achievable if individuals could be prevented from retiring except through ill-health; for in that case consumption would be a constant independent of age and state of health.

7. Private Saving

As in the simpler models, we can ask whether individuals would attempt to alter their consumption plans if they thought they could lend or borrow at the zero interest rate which the government faces. Calculating the gain from savings, we shall show that individuals would choose to save.

A consumer working at date t wants to lend for repayment at date t' if

$$u_1'(c_1(t)) < [u_1'(c_1(t'))(1 - F(t')) + \int_t^{t'} u_2'(c_2(s))f(s)ds]/[1 - F(t)]$$

When the opposite inequality holds, the consumer would choose to lend for repayment at date t'.

Dividing by $t' - t$ and letting $t' \to t$, we find that the consumer wants to save if

$$\frac{d}{dt}u_1'(c_1) > \frac{f}{1 - F}(u_1' - u_2') \tag{83}$$

Only if there were equality here would the individual be content with the saving being done on his behalf by the government. Reference to Theorem 3 shows that the optimal policy does not eliminate incentives to save or dissave; for in the optimum,

$$\frac{d}{dt}\left(\frac{1}{u_1'}\right) = \frac{f}{1-F}\left(\frac{1}{u_1'} - \frac{1}{u_2'}\right) \tag{84}$$

Now a simple calculation shows that

$$-(u_1')^2\left(\frac{1}{u_1'} - \frac{1}{u_2'}\right) = \frac{(u_1' - u_2')^2}{u_2'} + u_1' - u_2'$$

$$> u_1' - u_2' \tag{85}$$

Applying this inequality to (84), we obtain

$$\frac{d}{dt}u_1' > \frac{f}{1-F}(u_1' - u_2') \tag{86}$$

This proves that the consumer wants to lend, that is, to save. The government can prevent this happening by suitable taxation of saving.

It is interesting to enquire what the government should do if it has chosen, or is constrained, to allow a perfect untaxed capital market. Then in equilibrium

$$\frac{d}{dt}u_1' = \frac{f}{1-F}(u_1' - u_2') \tag{87}$$

and it is also the case that

$$u_1' = u_2' \quad \text{at} \quad r. \tag{88}$$

This last equality holds because of the possibility of saving for retirement, or borrowing against retirement benefits from the period when one knows one will be retired.

We conjecture that the third-best optimum, when private saving is unconstrained, is defined by (87), (88) and the condition that the consumer be indifferent about retiring age. We have not been able to prove this.

8. The Magnitude of Transfers

An insurance scheme is said to be actuarially fair if variations in the date of retirement have no effect upon the net discounted transfers to the individual. The optimal social insurance scheme we have derived is not in this sense actuarially fair. The net discounted transfer to an individual retiring at s is, in our notation, $z(s)$ (see equation (33)).

Theorem 5 For $s < r$, z is a decreasing function of s when social insurance is optimal. That is, those forced to retire earlier receive larger net transfers from the government.

Proof Since

$$z(s) = \int_0^s c_1(t)dt + c_2(s)(T - s) - s,$$

$$z'(s) = c_1(s) - c_2(s) + \dot{c}_2(s)(T - s) - 1,$$

$$= c_1 - c_2 + \frac{u_2 - u_1}{u_2'} - 1 \qquad (89)$$

By Theorem 4, the right hand side of (89) is zero at the optimal r, and negative on the $u_1' = u_2'$ curve to the left of the terminal point. The situation is shown in Figure 7, where the curve labelled $z' = 0$, which is the locus of (c_1, c_2) at which the right hand side of (89) vanishes, is shown sloping downwards and to the right from the terminal point of the optimal path. To justify this, we can differentiate the right hand side of (89) with respect to c_2, obtaining $-(u_2 - u_1)u_2''/(u_2')^2$. This is positive above the curve $u_1 = u_2$. Consequently points between the curves $u_1' = u_2'$ and

$u_1 = u_2$ for which c_1 is less than $c_1(r)$ all have $z'(s) < 0$. Since the optimal path lies entirely in this region, the theorem is proved.

The meaning of this result is that those who are unfortunate enough to suffer disability early in life receive a larger net transfer from the State than those able to work until late in life. The optimal social insurance scheme subsidises those who retire early, though only to the extent compatible with maintaining incentives to work.

The following result is of interest.

Corollary to Theorem 5 Under the optimal policy, for all $t \leq r$,

$$c_1(t) - c_2(t) < 1,$$

i.e. the extra consumption obtained by working is always less than the marginal productivity of labour.

Proof By the Theorem, the right hand side of (89) is negative. We have also shown that $u_2 > u_1$ on the optimal path. Hence $c_1 - c_2 - 1 < 0$, as was claimed.

9. Social Security in the United States

Despite the level of mathematical complexity, the models we have considered are very special. It would be silly to base policy on the particular equations we have derived. There are two aspects of the current U.S. publicly provided pensions which it may not be premature to criticize on the basis of the analysis we have performed.

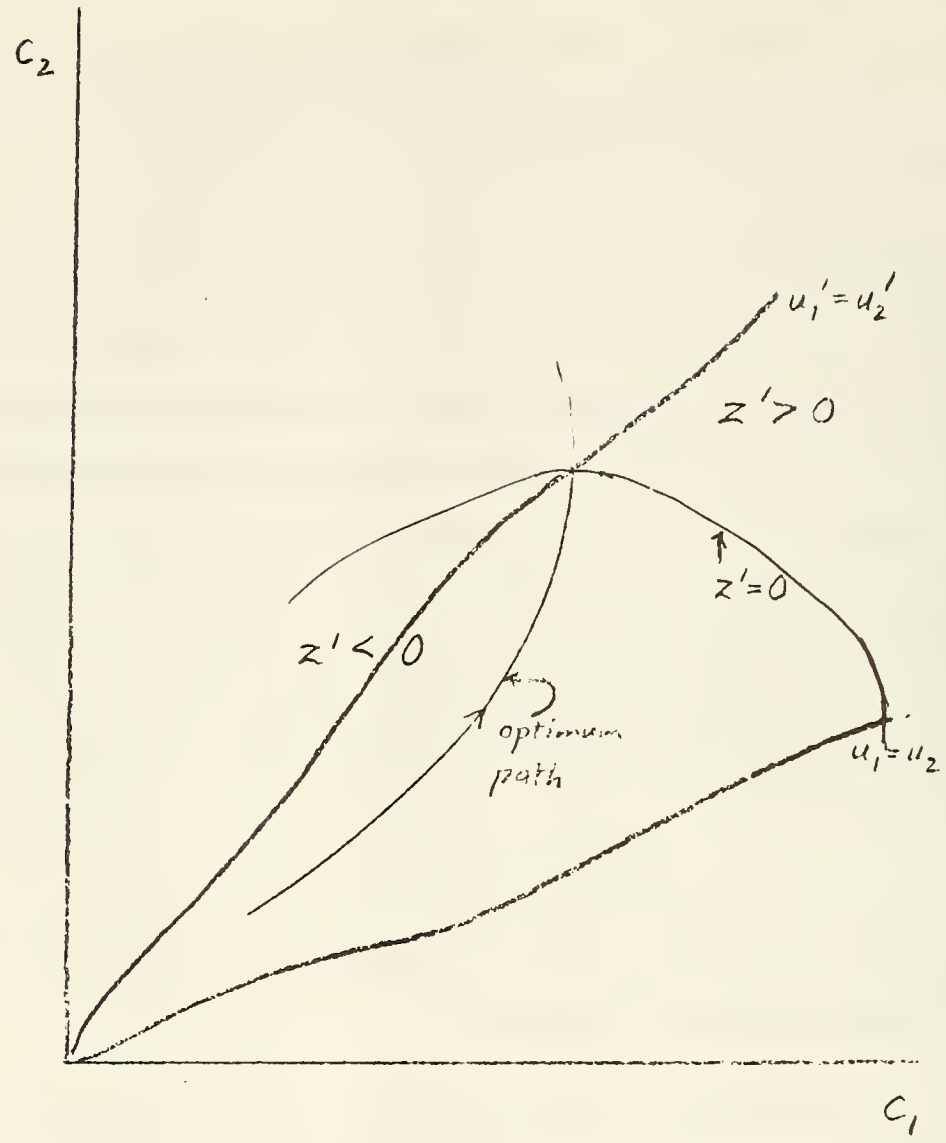

Figure 7

First we shall put the models briefly in perspective. If individuals are saving rationally, the incentive for work comes from the change in their lifetime budget constraints with additional work. If they are consuming their net wage, it is necessary to consider separately the incentives which come from current wages for work and from increased future pensions as a result of additional work. If they are following savings rules which differ from both of these models, the two parts of compensation matter differently, although not necessarily in the manner we have analysed. The U.S. population no doubt contains individuals whose behaviour is describable by a wide variety of models, and not just the fully rational model.

In the model analysed we found that a growing pension benefit permitted a higher pension, relative to the wage, than would be possible otherwise given the moral hazard problem. However if a pension which grows with work done is to serve as an incentive for work, individuals need to be aware of the relationship between pension and work done. While it would be expensive, a greater flow of information from the Social Security administration to individuals nearing retirement age about their own pension prospects might well be worth the cost.

Under the current U.S. system individuals receive their pension independent of whether they continue working once they reach age 72. Before that age there is an earnings limitation on pension eligibility. In terms of our notation the payment of benefits independent of retirement represents a large and discontinuous increase in c_1, the consumption enjoyed while working. In our analysis we found that a growing level of c_1 was optimal but that the growth should be continuous. c_1 can be increased continuously

either by reducing taxes repeatedly for those continuing to work or by paying a steadily growing fraction of pension benefits independent of any retirement test. For example, 15% of benefits might be paid at age 66 independent of retirement and 85% subject to the retirement test. The former fraction would grow steadily to 100% as in individual ages to 72.

10. Conclusions

The main results we have obtained may be summarised by highlighting what they would recommend if they were applicable to the real world. Many of the results suggest practicable policy changes, and may be worthy of study in more realistic models.

(1) The presence of moral hazard implies the desirability of policies that leave consumers indifferent about the date of retirement. This conclusion might be modified in models where individuals begin with different abilities, or different probability distributions for the loss of earnings.

(2) It is an essential adjunct to an optimal insurance scheme that capital transactions by consumers be taxed. In most cases, this would be a positive tax, discouraging saving. This conclusion might be modified if we allowed for nonrational saving behaviour.

(3) Post-retirement benefits show a strong tendency to increase with the age of retirement.

(4) Contributions to the insurance scheme should diminish with age, sometimes quite rapidly. They may become negative eventually. This is not as odd as it seems, for the optimal

allocation can be accomplished alternatively by paying part of benefits independent of retirement. The Corollary to Theorem 5 shows that the additional consumption from working rather than retiring immediately should never exceed the marginal product.

(5) There are some indications that contributions diminish more rapidly, and benefits increase more rapidly, as age increases.

(6) Under an optimal scheme, the incidence of retirement for reasons other than ill-health may be very low, even when the utility gain from healthy retirement is quite large. Despite this, the form of the optimal scheme may be very different from a first-best redistribution between people of different ages that ignores all incentive considerations.

(7) An actuarially fair scheme may discourage retirement less than an optimal scheme.

Footnotes

1. Financial support by the N.S.F. during the preparation of this paper is gratefully acknowledged. Computing assistance was provided by the Computing and Research Support Unit of the Oxford University Social Studies Faculty.

2. In another paper, we consider a diverse population, whose savings opportunities are limited only by linear taxation.

3. The theoretical effects of social insurance on retirement have been studied by Feldstein (1974) and Sheshinski (1977).

4. The model under consideration is equivalent to the standard maximization of an additive social welfare function. The particular distribution of characteristics assumed here has not, to our knowledge, been investigated previously.

5. For example, by preserving utility and marginal utility at c_1^* while increasing risk aversion. By analogy to Figure 3, the curve $u_1 = u_2$ is shifted up.

6. R equals the inverse of one plus the interest rate.

7. In addition it will be necessary to restate the moral hazard constraint to allow for the combined choice of savings and labour plans.

8. To achieve the optimum at D in Figure 4, the government must either close the capital market, or introduce nonlinear wealth taxation to ensure that individuals do not desire to save.

9. The analysis of optimal r is lengthy and rather technical. The impatient reader can turn to Theorem 4 at the end of this section, which states the essential results. The length and delicacy of the argument arises partly from the difficulty of establishing that maximum utility is a differentiable function of r - if it were not, unpleasant possibilities would arise - and partly from the difficulty of discussing the case $r = T$ directly. The surprising feature is that, under plausible assumptions, a simple first-order condition for optimal r can be obtained, and defines optimal r uniquely.

10. We would not expect this result without the assumption that the marginal utility of consumption of a nonworker is independent of his health.

11. To put it another way, higher retirement benefits are an incentive to work in every earlier period, the later the date the more periods in which work is encouraged.

12. If the utility function with labour y were $\log c + \log (2 - y)$, implying unit labour supply in the absence of lump sum income, the disutility of work is $\log (2 - 0) - \log (2 - 1) = \log 2 = .6931$ This Cobb-Douglas model probably overstates the utility of not working at all.

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